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B.Sc (Physics) Part-III, Paper-V, Group-A

sunilphy30@gmail.com

Tensor Part-IV

Equality & null tensor: consider two tensors

$A^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ and $B^{i_1 \cdots i_p}_{j_1 \cdots j_q}$.

If both the tensors have same contravariant and covariant rank and every component of one is equal to the corresponding component of the other, then

$$A^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} = B^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} \quad \text{--- (1)}$$

The two tensors have the same contravariant and covariant rank, they are called of the same type.

null tensor:- If the N^k components of a tensor of total rank k identically vanish, we say it to be a null tensor

Addition and substraction of tensor:-

Two tensor of the same type can be added and substracted

The resultant tensor will have the same rank as the original tensors.

$$C^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} = A^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} + B^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} \quad \text{--- (2)}$$

One can prove that $C^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ is a tensor. Let us write the transformation components of $B^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ in the form

$$\bar{B}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} B^{i_1 \cdots i_p}_{l_1 \cdots l_q} \quad \text{---} (3)$$

Similarly

$$\bar{A}^{\alpha_1 \alpha_2 \cdots \alpha_p}_{\beta_1 \beta_2 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} A^{i_1 i_2 \cdots i_p}_{l_1 l_2 \cdots l_q} \quad \text{---} (4)$$

Adding (3) and (4)

$$\bar{A}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} + \bar{B}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} \left(A^{i_1 \cdots i_p}_{l_1 \cdots l_q} + B^{i_1 \cdots i_p}_{l_1 \cdots l_q}\right) \quad \text{---} (5)$$

Now writing the sum of the components of the two tensors in the barred coordinate system as

$$\bar{C}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \bar{A}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} + \bar{B}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}. \quad \text{---} (6)$$

Now from Eq. (5) + (6)

$$\bar{C}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} C^{i_1 \cdots i_p}_{l_1 \cdots l_q} \quad \text{---} (7)$$

From Eq. (7) it is obvious that $C^{i_1 \cdots i_p}_{l_1 \cdots l_q}$ is a tensor of contravariant rank p and covariant rank q.

Similary we can define substraction of the two tensors $A^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ and $B^{i_1 \cdots i_p}_{j_1 \cdots j_q}$. ~~Let Substra~~

$$A^{i_1 \cdots i_p}_{j_1 \cdots j_q} - B^{i_1 \cdots i_p}_{j_1 \cdots j_q} = D^{i_1 \cdots i_p}_{j_1 \cdots j_q} \quad \text{---} (8)$$

Outer product:

Let A^{ij}_k and B^p_q are two tensors and A^{ij}_k has total rank three and, therefore, has N^3 components. B^p_q has total rank two and N^2 components.

∴ Each component of one tensor is multiplied by every component of the other. The resulting set of quantities gives a tensor whose rank is the sum of the ranks of the two original tensors.

We write the transformation equations for A^{ij}_k and B^p_q as

$$\bar{A}^{\alpha\beta}_{\gamma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} A^{ij}_{k}, \quad \text{--- (9)}$$

$$\bar{B}^{\rho}_{\sigma} = \frac{\partial \bar{x}^{\rho}}{\partial x^{p}} \frac{\partial x^{q}}{\partial \bar{x}^{\sigma}} B^{p}_{q} \quad \text{--- (10)}$$

Now

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\rho}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial \bar{x}^{\rho}}{\partial x^{p}} \frac{\partial x^{q}}{\partial \bar{x}^{\sigma}} A^{ij}_{k} B^{p}_{q} \quad \text{--- (11)}$$

Let us define $C^{ijp}_{kq} = A^{ij}_{k} B^{p}_{q}$, $\bar{C}^{\alpha\beta\rho}_{\gamma\sigma} = \bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\rho}_{\sigma}$. --- (12)

∴ Next, we can write Eq. (11) as

$$\bar{C}^{\alpha\beta\rho}_{\gamma\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\rho}}{\partial x^{p}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial x^{q}}{\partial \bar{x}^{\sigma}} C^{ijp}_{kq} \quad \text{--- (13)}$$

tensor of contravariant rank 3 and covariant rank 2;

C^{ijp}_{kq} — has total rank 5 and hence N^5 components, each of which ~~is the~~ is the product of one component A^{ij}_k with one B^p_q.

Eq. (13) defines the outer product or Kronecker product of two tensors. This can also be extended to more than two tensors.